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THE ROYAL SOCIETY

Some aspects of flow of granular materials in hoppers

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The main part of the paper is devoted to the description of the localized deformation zones in plane (slow) flow of granular materials from hoppers. Specifically, two types of zones are discussed in detail: material layers, where the particles undergo deformation without crossing the band, and shocks, where the deformation takes place when the particles cross the band. The fundamental requirements and assumptions underlying the description of the two types of localized deformation are analysed. Examples of quasi-static solutions within the framework of plasticity theories are presented, which demonstrate the possibility of predicting the type of hopper flow (mass/funnel), and the periodic occurrence of localized deformation zones. Also, the necessary conditions for a stationary shock taking into account the inertia of particles crossing the band are derived. It is demonstrated that some hardening is required for a stationary shock to exist. Lastly, the relevance of an appropriate constitutive model of granular material for describing the formation of arches and empty channels which obstruct continuous gravitational flow is addressed.

Keywords: granular materials; hoppers; shear bands; shocks

1. Introduction

Flow of granular materials through storage vessels, and hoppers in particular, has been the topic of extensive experimental and theoretical research for several decades. The interest stems from problems encountered in bulk material handling operations, as well as the fact that the theoretical treatment of flow in hoppers provides a class of benchmark boundary-value problems for analysing and predicting the behaviour of granular materials under external mechanical excitations. In fact, the inherent symmetry of wedge or conical hoppers allows reduction of the number of spatial variables to two or even one, thereby reducing the number of equations which govern the behaviour of the material during flow. This, in turn, often permits for solving these equations analytically, and for comparing the results with physical experiments or numerical simulations to assess their validity. Accordingly, a significant number of publications have been devoted to this topic, with the results summarized in bulletins (Jenike 1964*a*), monographs (Drescher 1991; Nedderman 1992), and review articles (Nedderman *et al*. 1982; Tüzün *et al*. 1982), to name a few.

When considering flow through hoppers, it is generally assumed that collisional effects characteristic of rapid flow in chutes can be neglected. Also, the influence of interstitial gas on particle flow is usually regarded as secondary in comparison to particle interaction, which allows us to consider one-phase material flow. With these simplifications, either the methods of continuum mechanics, or the methods

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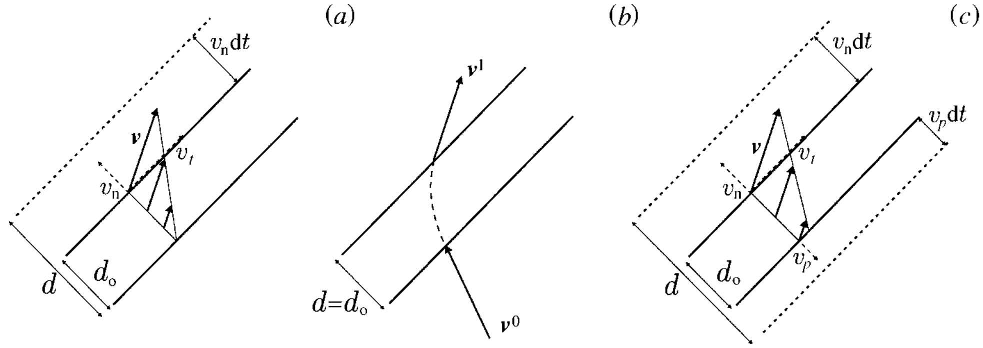

Figure 1. Shear bands: (*a*) material layer, (*b*) shock, (*c*) material layer/shock.

considering only the kinematics of particles, form the framework of the majority of theoretical analyses. In the following, the first framework will be employed, and emphasis will be placed on some aspects of flow that remain challenging or somewhat controversial. More specifically, we will concentrate on non-homogeneities in the velocity field developing during discharge from hoppers, and on impediments to continuous flow.

2. Non-homogeneities in flow

It has long been recognized that one of the characteristic features of mass flow of uniformly packed, rough faceted granular materials in plane (wedge-type) hoppers is the occurrence of zones of localized deformation in an otherwise continuous field of flow. These have been given the name shear bands, rupture zones or (strong) velocity discontinuity lines, and they can be detected using various measuring techniques, notably stereophotogrammetry and radiography. In spite of being frequently reported, the mechanics of the formation and propagation of these features remains poorly understood, and only a limited number of theoretical analyses have been performed to explain and predict their occurrence (Cutress & Pulfer 1967; Pariseau 1969/70; Drescher *et al*. 1978; Drescher & Michalowski 1984; Michalowski 1984, 1987, 1989, 1990; Pitman 1986). This is in contrast to the vast literature on localized deformation in geotechnical problems, where the bifurcation theory of nonlinear continua has generally been accepted as the theoretical foundation, with ample experimental validation and growing implementation in solving boundary-value problems, as detailed in the monograph by Vardoulakis & Sulem (1995).

The success in understanding and description of the localized deformation in geotechnical boundary-value problems can be attributed to the fact that most belong to the class of incipient flow problems. A gradual increase in boundary displacements resulting from specified velocities, often discontinuous, makes it possible to capture the conditions for the onset of localization, with the zone of localized deformation regarded as a material layer. In the case of flow in hoppers, however, localized deformation appears weakly related to the boundary velocities, and it is not always clear as whether the zone is a material layer or a shock.

The concept of the shear band as a material layer implies a constant mass of material simultaneously undergoing the same deformation, as no material leaves or enters the band. If the material dilates or compacts its density changes, and so does the thickness of the band (figure 1*a*). On the other hand, if the shear band is regarded as a shock, the same mass of material enters and leaves the band at any instance of time, and material elements undergo gradual deformation when crossing the band. This results in density change across the band if the material is dilating or compacting, with the thickness of the band remaining constant during flow (figure 1*b*). As, in general, the thickness of the material layer and the shock may be similar, and if the photographic or radiographic measurements are taken over a short time interval, the overall appearance of these two fundamentally different features may be similar when inferred from stereopairs or radiographs (Drescher & Michalowski 1984). It also is possible that the material outside a sheared material layer is subjected to erosion due to particle interlocking with the boundaries of the layer moving as shocks (figure 1*c*), which leads to mixed, material/shock type bands, and this complicates further the interpretation of the measurements (Han & Drescher 1993).

The essential element in the analysis of localized deformation bands as shocks is the requirement for the velocities and densities on both sides of the band to satisfy the continuity equation derived from the mass conservation law. For a non-moving shock, the continuity equation can be written as

$$\rho^1 v_{\mathrm{n}}^1 = \rho^0 v_{\mathrm{n}}^0 \tag{2.1}$$

or as

$$[\rho v_{\mathrm{n}}] = 0, \tag{2.2}$$

where ρ is the density, v_{n} is the normal component of the velocity v_i, and $[\cdot]$ denotes the jump in a given quantity; the superscripts '0' and '1' refer to the side ahead and behind the shock, respectively. As full equivalence exists between flow of material through a shock or shock moving through the material, equations (2.1) and (2.2) also hold if the velocity v_i is measured with respect to a shock moving with the speed of propagation v_i^{p}. Alternatively, with the velocities measured with respect to a stationary system, equation (2.1) becomes

$$\rho^1 (v_{\mathrm{n}}^1 - v_{\mathrm{n}}^{\mathrm{p}}) = \rho^0 (v_{\mathrm{n}}^0 - v_{\mathrm{n}}^{\mathrm{p}}). \tag{2.3}$$

The first attempts to investigate bands of localized deformation in hoppers hinge on constructing the kinematic counterpart to the (quasi-) static solution for a rigid-perfectly plastic, pressure dependent (Mohr–Coulomb or Drucker–Prager) model of a granular material. This is done by tracing the velocity characteristics net related to the stress field through the flow rule. As the equations governing the kinematics are linear, strong velocity discontinuities coinciding with the velocity characteristics are permissible. The location of discontinuities, and the magnitude of velocity jumps, can then be determined if the stress field is continuous and the velocity boundary conditions are known. These are seldom uniquely defined, however, and additional assumptions are necessary to arrive at a solution. This is because the flow of granular materials in hoppers is usually driven by gravity and not by prescribed movement of a boundary, as is the case of extrusion through wedge-shaped dies. Accordingly, the solutions obtained are used for explaining experimental observations rather than for predicting the actual pattern or type of flow. This is illustrated clearly in papers by Pariseau (1969/70) and Michalowski (1984, 1987), with the examples of characteristics net and velocity discontinuities shown in figure 2. A direct numerical integration of the governing static and kinematic equations has also been performed (Pitman

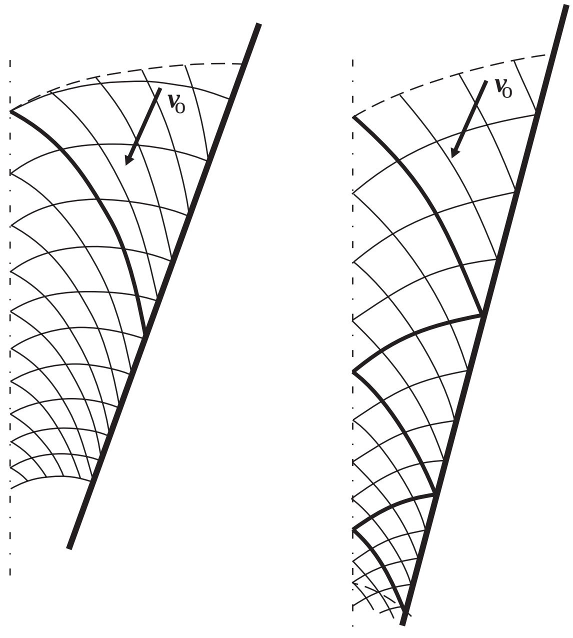

Figure 2. Velocity characteristics and velocity discontinuities (Michalowski 1984, 1987).

1986), and shows strong inhomogeneities in the velocity field which can be interpreted as the occurrence of localized deformation.

It should be noticed that even though these solutions aim at describing advanced steady flow, the way they are constructed actually corresponds to incipient flow if the material displays volume changes during shear. In fact, no use is made of equation (2.2) in relating the density of the material to the velocity jumps across the discontinuities, and this is reflected in assuming constant bulk unit weight of the material in constructing the stress field; only when the material is incompressible is the continuity equation satisfied identically everywhere.

A notable exception, when the solution of the velocity field serves as a criterion for predicting the type of flow, is the radial (similarity) solution for incompressible material considered by Jenike (1964*b*, 1987). This author has demonstrated that in conical hoppers there exists a critical combination of material effective friction angle ϕ_{ef}, wall friction angle φ_{w}, and hopper half-included angle θ_{w} beyond which the radial field cannot be constructed. This critical combination was taken as a criterion limiting the occurrence of mass flow; there is no such a limit, however, for a plane radial flow.

A mass/funnel flow criterion derived directly from the orientation of shear bands in hoppers has been proposed by Drescher (1992). Experimental observations of the onset of flow indicate that in mass flow the shear bands emanating from the edges of an outlet propagate towards the opposing walls and undergo successive reflections (figure 3). In funnel flow, on the other hand, the shear bands propagate towards the adjacent walls. The vertical orientation of the bands, corresponding to plug flow, then is regarded as separating mass and funnel flow. The analysis of the limiting plastic state of stress at the wall next to the outlet leads to the following criterion

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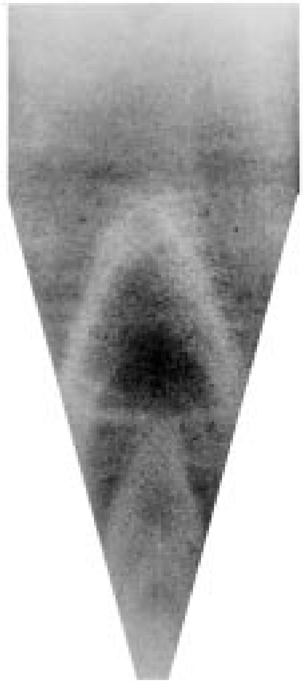

Figure 3. Radiograph of reflecting shear bands.

that must be satisfied for mass flow,

$$\tan \varphi_{\mathrm{w}} \leqslant \frac{\sin \phi \cos(2\theta_{\mathrm{w}} - \psi)}{1 + \sin \phi \sin(2\theta_{\mathrm{w}} - \psi)}, \tag{2.4}$$

where ϕ is the friction angle at shear band initiation, and ψ is the corresponding dilatancy angle; the results are shown in figure 4 for $\psi = 2/3\phi$. In deriving equation (2.4) it was assumed that the shear band coincides with the velocity characteristic. Equation (2.4) can be modified further, by aligning the shear band along the direction resulting from the bifurcation analysis which deviates from the direction of a characteristic by $(\phi - \psi)/4$ (Vardoulakis & Sulem 1995); this leads to replacing ψ in (2.4) by $(\phi + \psi)/2$.

Plug flow experiments also indicate that with progressing discharge, the vertical shear bands tend to move laterally outward thus widening the region of flow. This is seen in radiographs indicating clearly a boundary between the dense and dilated material (figure 5). As the movement of the boundary seems to be non-periodic, with the new material continuously entering the region of flow, the band cannot be classified as a material layer but as a propagating shock. Applying the continuity condition (2.3) we conclude that particles entering the region of flow must possess a velocity component directed towards the hopper's symmetry line. However, no accurate data on particle velocities are available to illustrate this conclusion, and no analyses have been performed to describe the corresponding velocity field analytically.

The next approach, presented by Michalowski (1989, 1990), derives from the kinematic method of plastic limit analysis. A kinematically admissible velocity field with straight inextensible bands of localized deformation with constant velocity gradi-

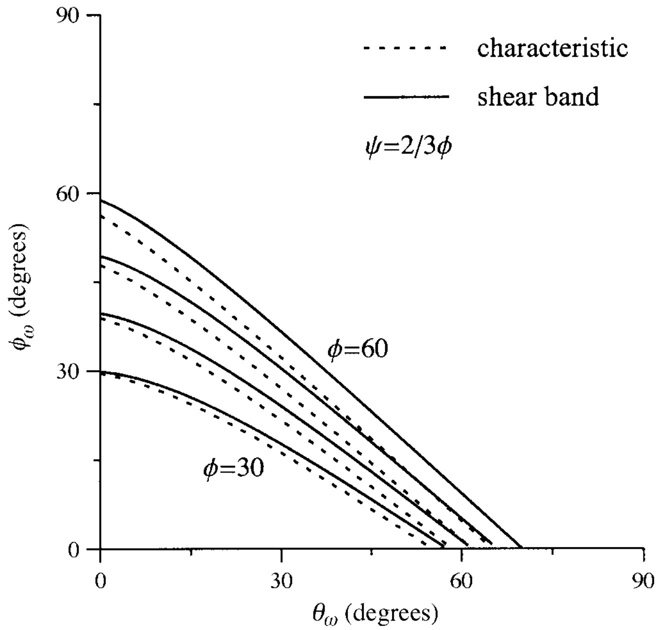

Figure 4. Regions of mass/funnel flow.

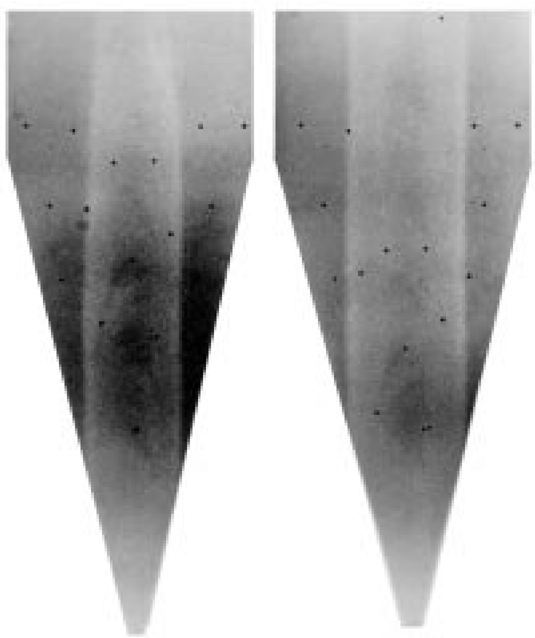

Figure 5. Radiographs of moving shocks in plug flow.

ent is postulated as an approximation of fields often observed in experiments (figure 6*a*). The flowing material is assumed to be two-dimensional, rigid-plastic hardening/softening. The location and orientation of shear bands is found from minimizing

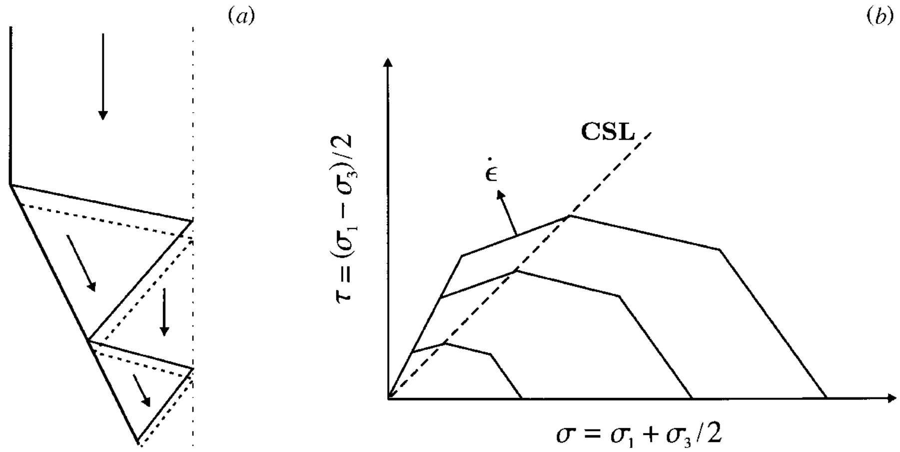

Figure 6. (*a*) Velocity discontinuities in a bin/hopper; (*b*) evolution of the yield condition (Michalowski 1989, 1990).

the rate of energy dissipated on deformation along the band, calculated from the energy dissipation per unit length of the band of thickness d given by

$$\dot{D} = \int_0^d (\sigma\dot{\epsilon}_\sigma + \tau\dot{\epsilon}_\tau)\,\mathrm{d}n, \tag{2.5}$$

where the strain rates are

$$\dot{\epsilon}_\sigma = \frac{[v]\tan\psi}{d}, \qquad \dot{\epsilon}_\tau = \frac{[v]}{d}, \tag{2.6}$$

and σ and τ are the mean and deviator stresses, respectively. The stresses satisfy a piecewise linear yield condition resembling that of the modified Cam clay model (figure 6*b*), with the density ρ as a hardening parameter varying across the band according to

$$\rho = \rho^0\left(1 + \tan\psi\frac{[v]}{v_\mathrm{n}^0}\frac{n}{d}\right)^{-1}, \tag{2.7}$$

and with the dilatancy angle ψ defined by the associative flow rule. For a stationary shock, when material elements cross the band, $\dot{D}$ is independent of the band thickness d. If, however, the band is allowed to move downward, the rate of energy dissipation depends on the thickness of the band through the time interval during which the element remains within the moving band. It then follows that a different initial band thickness d leads to a different rate of energy dissipation in the course of flow. Note that in all cases, the least energy is dissipated when the material dilates and softens. Figure 7 shows a comparison of energy dissipation for a stationary and a moving uppermost band as a function of time, normalized by the velocity modulus $|v^0|$ and initial band length b. It is seen that for a sufficiently thin band, the rate of energy dissipation when the band is moving is less than when the band is stationary, and this can be interpreted as likelihood of the formation of material or mixed type shear bands in flow of granular materials undergoing softening during shear. Periodicity of band formation can also be deduced from this analysis, and this seems to be supported by experimental results showing a number of closely spaced bands (figure 8).

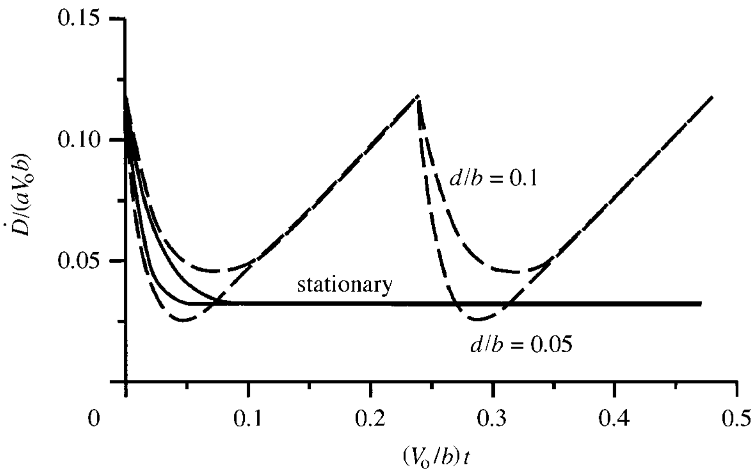

Figure 7. Energy dissipation as a function of time (Michalowski 1989).

Figure 8. Radiograph of periodic shear bands.

Another approach to analysing localized deformation, under development by Garagash *et al*. (1998), seeks the necessary conditions on the existence of a shock while accounting for the inertia during flow. The material is modelled as two-dimensional, rigid-plastic hardening/softening, with the Mohr–Coulomb yield condition and potential flow rule. The friction angle ϕ and the dilatancy angle ψ are taken as functions

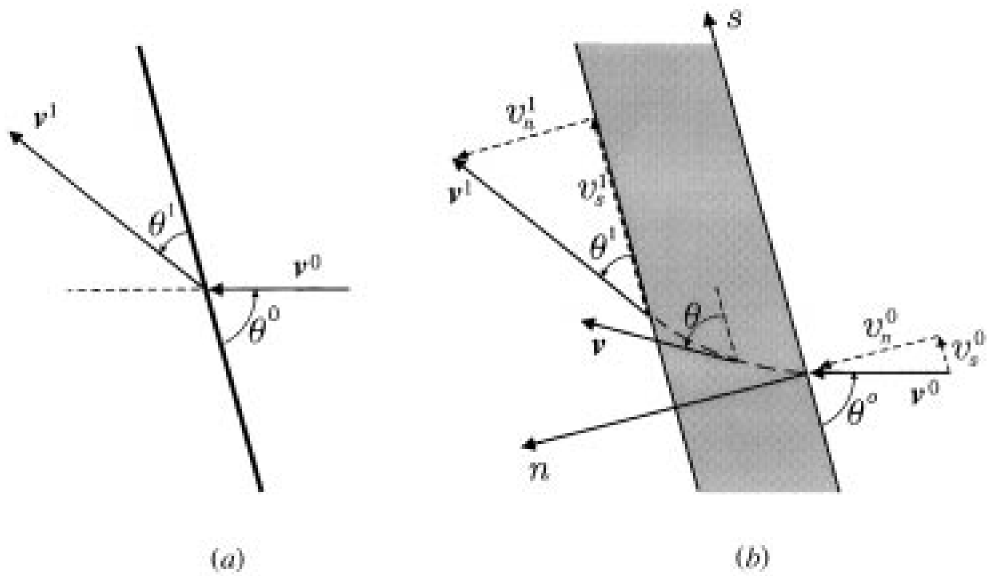

Figure 9. Velocity jump across a stationary shock.

of the effective strain γ defined as

$$\gamma = \int_0^t D\,\mathrm{d}t, \tag{2.8}$$

where D is the shear intensity of the deformation-rate tensor D_{ij}. Referring to figure 9, the velocity components v_n and v_s within an inextensible and stationary shock ($\partial/\partial s = 0$, $\partial/\partial t = 0$) can then be expressed as

$$v_\mathrm{n} = v_\mathrm{n}^0 \delta_\mathrm{n}, \qquad v_\mathrm{s} = v_\mathrm{s}^0 + v_\mathrm{n}^0 \delta_\mathrm{s}, \tag{2.9}$$

where

$$\delta_\mathrm{n} = \exp\left(\int_0^\gamma \sin\psi\,\mathrm{d}\gamma\right), \qquad \delta_\mathrm{s} = \int_0^\gamma \delta_\mathrm{n}\cos\psi\,\mathrm{d}\gamma. \tag{2.10}$$

Making use of (2.1), we can write the momentum balance law across a shock as

$$\rho^0 v_\mathrm{n}^0 [v_\mathrm{n}] = [\sigma_\mathrm{nn}], \tag{2.11}$$

$$\rho^0 v_\mathrm{n}^0 [v_\mathrm{s}] = [\sigma_\mathrm{sn}]. \tag{2.12}$$

With the help of the yield condition and flow rule, σ_nn and σ_sn can be expressed as

$$\sigma_\mathrm{nn} = \sigma(1 - \sin\phi\sin\psi), \tag{2.13}$$

$$\sigma_\mathrm{sn} = -\sigma\sin\phi\cos\psi, \tag{2.14}$$

where σ is the mean stress. Introducing the mean stress Σ normalized by the specific kinetic energy

$$\Sigma = \frac{\sigma}{\rho^0 |v^0|^2} \tag{2.15}$$

we can write equations (2.11) and (2.12) as

$$\delta_\mathrm{s}^1 \sin^2\theta^0 = -[\Sigma\sin\phi\cos\psi], \tag{2.16}$$

$$(\delta_\mathrm{n}^1 - 1)\sin^2\theta^0 = [\Sigma(1 - \sin\phi\sin\psi)], \tag{2.17}$$

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where θ^0 is defined in figure 9, and δ_s^1 and δ_s^1 correspond to $\gamma = \gamma^1$. Equations (2.16) and (2.17) relate the mean stress ahead of and behind the shock, Σ^0 and Σ^1, to the effective strain γ^1 quantifying the total deformation that takes place when the material crosses the shock. The solution to equations (2.16) and (2.17) depends on the form of functions $\phi(\gamma)$ and $\psi(\gamma)$, and boundary conditions. In some cases, however, it is possible to assess directly the character or the existence of the solution. For example, when both ϕ and ψ are constant we arrive from (2.16) and (2.17) at $[\Sigma] = 0$ and $\gamma^1 = 0$, which implies that no stress and velocity discontinuity is possible and, hence, no stationary shock can exist in a perfectly plastic material if the flow is considered as inertial.

When $\phi(\gamma) = \psi(\gamma)$ (associative material), equations (2.16) and (2.17) reduce to

$$\Sigma^0 \sin 2\phi^0 = 2\sin^2\theta^0\delta_s^1 + \Sigma^1 \sin 2\phi^1, \tag{2.18}$$

$$\sin^2\theta^0 \left\{(\delta_n^1 - 1)\sin\phi^0 + \delta_s^1\cos\phi^0\right\} = -\Sigma^1 \sin[\phi]\cos\phi^1. \tag{2.19}$$

As the left-hand side of (2.19) is always positive, and the mean stress is always compressive ($\Sigma < 0$), we conclude from (2.19) that a stationary shock may only exist if

$$[\phi] > 0, \tag{2.20}$$

i.e. when ϕ increases with γ and the material is hardening. The dependence of γ^1 and $\beta = \theta^0 - \theta^1$ on θ^0 for several values of Σ is shown in figure 10 for $\phi = \psi$ increasing nonlinearly with γ and reaching an asymptotic value; the dashed line corresponds to $\Sigma^1 \to -\infty$, i.e. to quasi-static flow. Note that for any angle θ^0 there are actually two admissible velocity jumps and two angles β (figure 11), and two sets of curves depicting the variation of β with θ^0, symmetric about $\theta^0 = \pi/2$, exist in figure 10; however, this is not shown for clarity of presentation.

In the general case of $\phi(\gamma) \neq \psi(\gamma)$ (non-associative material), equations (2.16) and (2.17) become

$$\Sigma^0(1 - \sin\phi^0\sin\psi^0) = \{\Sigma^1(1 - \sin\phi^1\sin\psi^1) - \sin^2\theta^0(\delta_n^1 - 1)\} \tag{2.21}$$

$$\sin^2\theta^0\left\{(\delta_n^1 - 1)\frac{\sin\phi^0\cos\psi^0}{1 - \sin\phi^0\sin\psi^0} + \delta_s^1\right\} = -\Sigma^1(1 - \sin\phi^1\sin\psi^1)\left[\frac{\sin\phi\cos\psi}{1 - \sin\phi\sin\psi}\right], \tag{2.22}$$

and a stationary shock may exist if

$$\left[\frac{\sin\phi\cos\psi}{1 - \sin\phi\sin\psi}\right] > 0. \tag{2.23}$$

The interpretation of condition (2.23) follows from a ϕ versus ψ diagram shown in figure 12, in which lines are drawn of several constant values of

$$a = (\sin\phi\cos\psi)/(1 - \sin\phi\sin\psi).$$

Given initial values of ϕ^0 and ψ^0, condition (2.23) is satisfied when ϕ^1 and ψ^1 locate at a higher value of a. It is seen that (2.23) is always satisfied when neither ϕ nor ψ are a decreasing function of γ, which characterizes loose granular materials undergoing hardening with deformation (solid line in figure 13). An example of a solution for both ϕ and ψ increasing with γ is shown in figure 14. Again, only one set of symmetric curves β versus θ^0 is shown in figure 14. If, however, ϕ and ψ

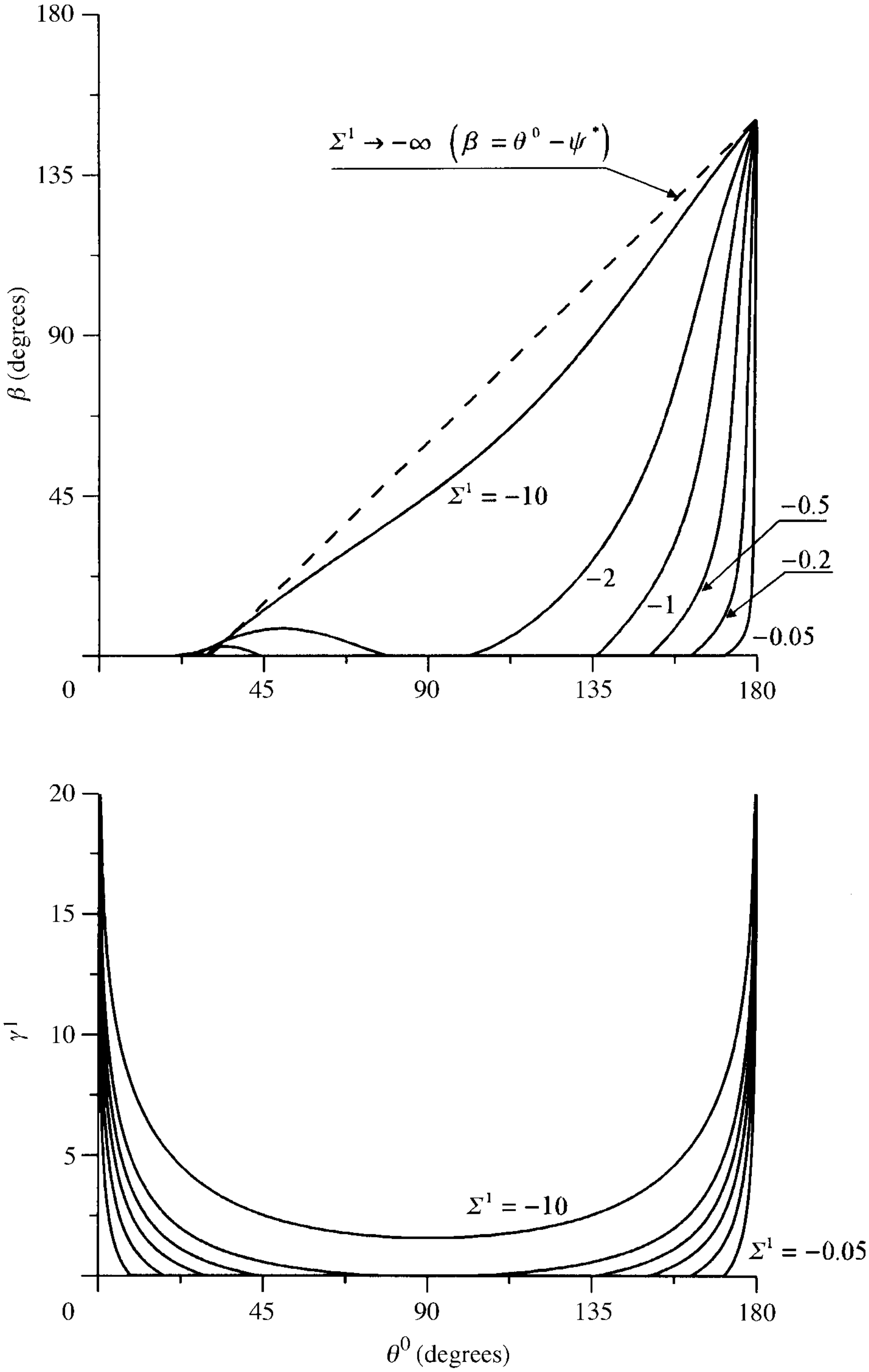

Figure 10. Solution for $\phi(\gamma) = \psi(\gamma)$ (associative hardening material).

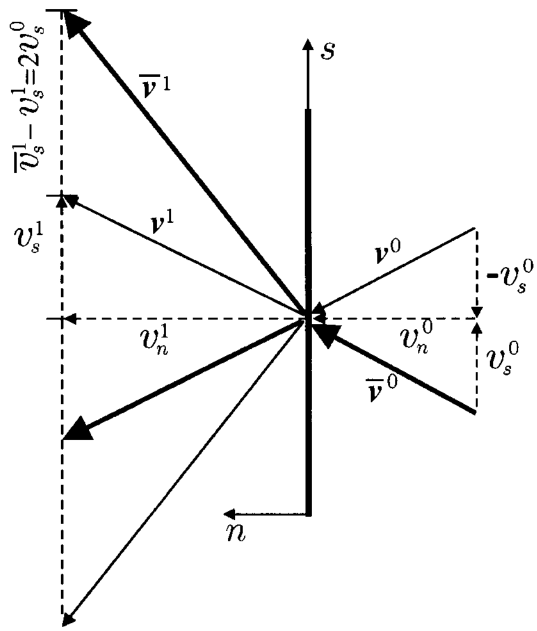

Figure 11. Multiple velocity jump solutions (associative material).

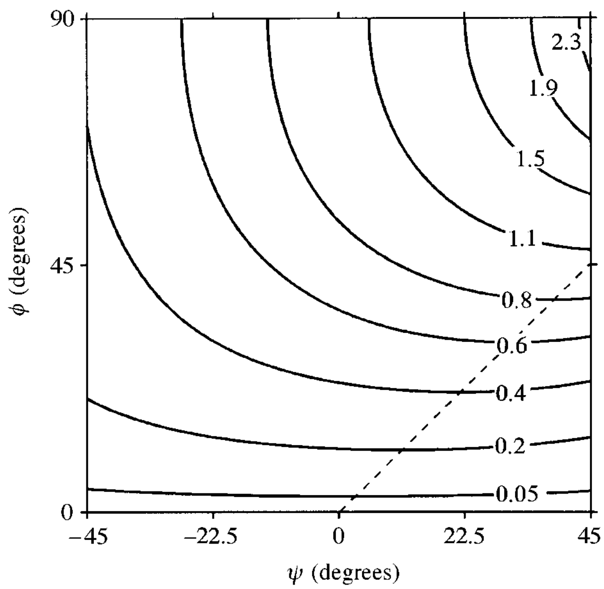

Figure 12. Isolines of constant $(\sin\phi\cos\psi)/(1-\sin\phi\sin\psi)$.

are increasing/decreasing functions of γ, which typify dense materials (dashed line in figure 13), a stationary shock may or may not be admissible, depending on the values of ϕ^1 and ψ^1 with respect to ϕ^0 and ψ^0. The solution for γ^1 and $\beta = \theta^0 - \theta^1$ as a function of θ^0 when condition (2.23) is satisfied differs from the case when both ϕ and ψ increase with γ only at small angle θ^0, and this is depicted in figure 15. The results above indicate that in medium dense and dense granular materials, stationary

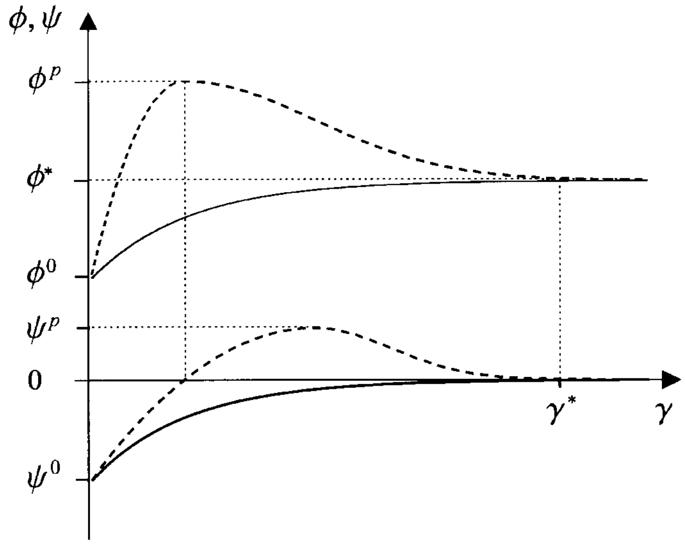

Figure 13. Function $\phi(\gamma)$ and $\psi(\gamma)$; solid line, loose material; dashed line, dense material.

shocks should not be observed if significant softening is possible during flow, as may be the case in mass flow. If the softening is moderate, however, a stationary shock is admissible, which seems to be illustrated by the propagating boundary of the flowing region in plug flow shown in figure 5, or the boundary between the central and two upper regions in mass flow shown in figure 16.

3. Impediments to flow

In the discussion of localized deformation during flow from hoppers, we have tacitly assumed that the granular material is free-flowing, which typifies uncemented coarse and medium coarse materials. With the decrease in particle size and with the increase of compaction, or due to the presence of agents such as moisture, various types of interparticle bonds may develop and result in particle cementation, with the material becoming non-free-flowing.

In non-free-flowing materials, self-supported arches or domes may form upon opening of the outlet or at some instance of flow. Continuous discharge may also be interrupted when an empty channel forms above the outlet, with the surrounding material remaining at rest. The reason for the formation of these impediments to flow is the ability of the material to sustain uniaxial or biaxial compression which develops next to the exposed surface of arches or channels. This fact, first recognized by Jenike & Leser (1963), has been the basis of several plasticity-based theoretical analyses aiming at predicting the size of an outlet preventing arching or channelling, as summarized, for example, by Drescher (1991).

Following Jenike & Leser (1963), most analyses of arching are based on comparing the strength of the material to the stresses that act in a stable arch or dome. This is shown schematically in figure 17*a*, where the variation along the hopper height of the (uniaxial) compressive strength denoted as σ_0, and of the maximum compressive stress in the arch denoted as σ_{1a}, are superimposed. The location where $\sigma_0 = \sigma_{1a}$ defines the minimum outlet size preventing arching. The stresses in the arch are

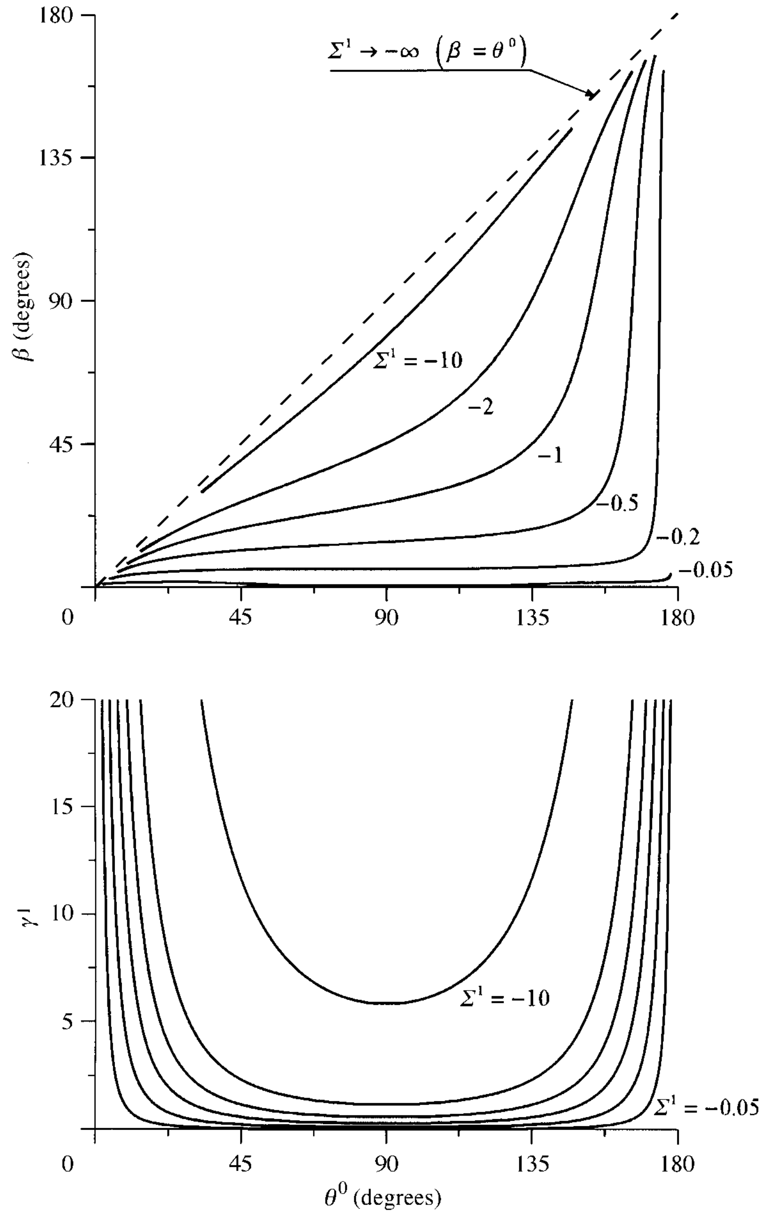

Figure 14. Solution for $\phi(\gamma) \neq \psi(\gamma)$ (non-associative hardening material).

derived from the analysis of statically determined structural members, which implies that no material parameters other than the unit weight and wall friction angle enter the derivation of the weight-induced stresses, and the latter are a linear function of the distance r. The compressive strength is determined from tests, and subsequently related to stresses σ_c that act during flow and consolidate the material. The consolidating stresses, again, are taken as a linear function of r, i.e. the radial solution using the local or the global equilibrium equations is postulated for the stresses.

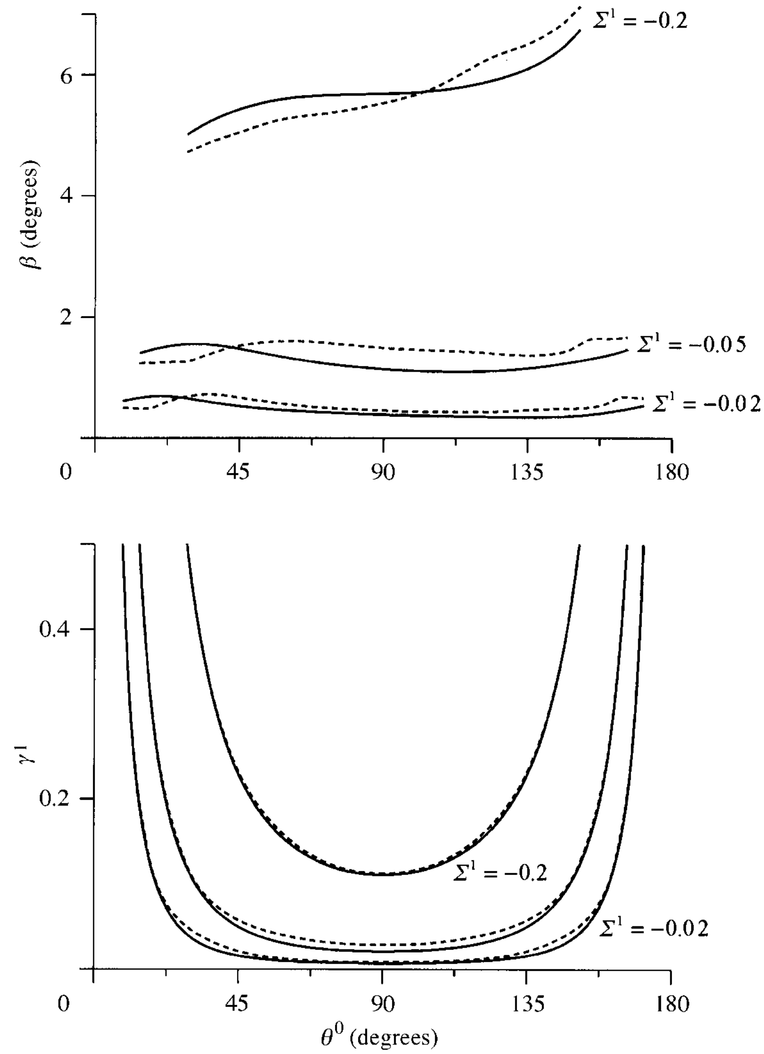

Figure 15. Comparison of solutions for $\phi(\gamma) \neq \psi(\gamma)$; solid line, hardening material; dashed line, hardening/softening material.

The underlying basis of material strength and flow stresses is the rigid-hardening/softening plastic model of Jenike & Shield (1959), conceptually similar to the well-known Cam clay model devoid of elasticity. The central element of these models is the ultimate or critical state, where the material displays purely frictional resistance, and this state is assumed for evaluating the stresses during flow. The instantaneous or current yield surface is a function of consolidating stress σ_c, and this defines the relation between σ_0 and σ_c termed the flow function FF (figure 17*b*). The consequence of this model is that $\sigma_0 = 0$ if $\sigma_c = 0$, i.e. the material has no strength if the consolidating stresses are null. This implies that the magnitude of σ_0 must approach the magnitude of σ_{1a} at $r = 0$ (figure 17*a*). Due to experimental difficulties in conducting tests at very low consolidating stresses that are typical for hoppers, the shape of the FF close to the origin is extrapolated from tests at higher stresses rather

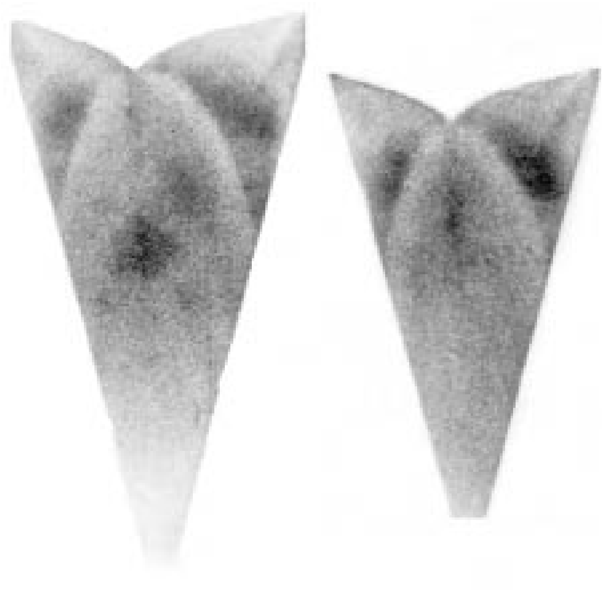

Figure 16. Radiograph of moving shocks in mass flow.

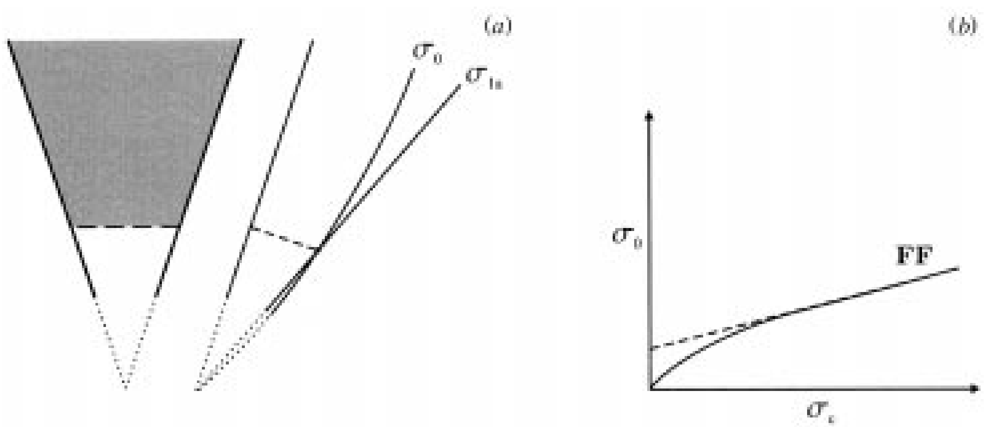

Figure 17. (*a*) Distribution of strength and arch-stresses in a hopper; (*b*) flow-function.

than determined directly. As discussed by Drescher *et al*. (1995*a*, *b*), extrapolations which lead to a non-zero compressive strength at $\sigma_c = 0$ (figure 16*b*) are inconsistent with the model assumed, for they violate the concept of a purely frictional critical state which, on the other hand, is the mandatory requirement for the existence of the radial solution for the flow stresses σ_c. These extrapolations are equivalent to postulating a consolidation-independent material cohesion or residual cohesion even at large shearing deformation. This concept, put forward by Molerus (1978), has not found application in modelling the behaviour of soils. It has been used by Enstad (1981), however, who arrived at the critical outlet size by considering global equilibrium of the material filling the hopper with no support from below. It appears that the issue of material strength at large shearing deformation characteristic for the flow in hoppers remains somewhat unresolved when analysing the impediments to flow.

4. Closing remarks

In discussing non-homogeneities in the flow of granular materials we have concentrated on plane or wedge-type hoppers. This is because detecting material shear bands or shocks in conical hoppers is technically difficult, and insufficient reliable experimental data exist for further theoretical studies. Although not well documented, there are also indications that in the flow through conical hoppers fewer zones of localized deformation form or are altogether absent. A qualitatively similar result has been obtained by Pitman (1986) in studying the convergence of the radial stress field in plane and conical hoppers; in conical hoppers the convergence is faster, and in plane hoppers oscillations in the mean stress are more apparent. As the velocity field is related to the stress field, this could be interpreted as a greater chance of localized deformations in plane hoppers. A greater instability in the evolution equations governing plastic flow in plane strain problems has also been discussed by Schaeffer (1987, 1990). We should mention here that strong velocity discontinuities may be inadmissible in some axisymmetric problems for rigid-perfectly plastic materials as illustrated by Shield (1955) and Cox *et al*. (1961), whereas they are always admissible in plane strain problems. All this points to possible significant differences in the mechanics of flow in conical and plane hoppers; unfortunately, most advanced studies have only considered the latter.

It seems that central in predicting possible formation of arches and empty channels in hoppers is the determination of an appropriate constitutive model of the material, which would resolve the issue of material strength (yield limit) as a function of consolidating stresses and large shearing deformation taking place during flow. This is related directly to necessary progress in measuring material response under small stress levels typical for hoppers.

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